Week 5

Recall: We introduced c-convexity, c-subdifferential, c-transform.

- $\varphi(x) + \varphi^c(y) + c(x,y) \geq 0 \quad \forall x \in X, y \in Y.$
- $\varphi(x) + \varphi^c(y) + c(x,y) = 0 \iff y \in \partial^c \varphi(x).$

Kantorovich duality $c : X \times Y \to \mathbb{R}$ continuous, bounded below.

If $\inf\limits_{\gamma \in \Gamma(\mu,\nu)} \int c \, d\gamma = (KP) < +\infty$, then

$$(KP) = \max_{\substack{\varphi, \psi : \\ \varphi(x)+\psi(y)+c(x,y) \geq 0}} \int_X -\varphi \, d\mu + \int_Y -\psi \, d\nu = (DP)$$

Remark: to show $\leq$ we actually prove that $\operatorname{supp} \bar\gamma$ c-cycl. monotone $\Rightarrow$

$$\exists \varphi \text{ c-convex} : \int c(x,y) \, d\bar\gamma(x,y) = \int -\varphi \, d\mu + \int -\varphi^c \, d\nu \leq (DP) = (KP).$$

We conclude that: $\bar\gamma$ has c-cyclic. monotone supp $\iff$ $\bar\gamma$ is optimal in (KP)

Corollary: Let c be continuous, bounded below. Then, it is equivalent:

- $\bar\gamma$ is optimal in (KP).
- $\operatorname{supp} \bar\gamma$ is c-cycl. monotone.
- $\exists \varphi$ c-convex s.t. $\operatorname{supp} \bar\gamma \subseteq \partial^c \varphi$

Corollary: if c l.s.c. bounded below, then $(KP) = (DP)$ still holds

(but maybe the max. in (DP) is a sup).

Proof: $\geq$ it is easy, from the same proof.

$\leq$ wlog $c \geq 0$. Take $c_n \geq 0$, $c_n \uparrow c$, c_n continuous.

From theorem,

$$\underbrace{\min_{\gamma \in \Gamma(\mu,\nu)} \int c_n \, d\gamma}_{\overset{!!}{=} \int c_n \, d\bar\gamma_n} = \sup_{\varphi(x)+\psi(y) \geq -c_n(x,y)} \int -\varphi \, d\mu + \int -\psi \, d\nu$$

$$\leq \sup_{\varphi(x)+\psi(y) \geq -c(x,y)} \int -\varphi \, d\mu + \int -\psi \, d\nu$$

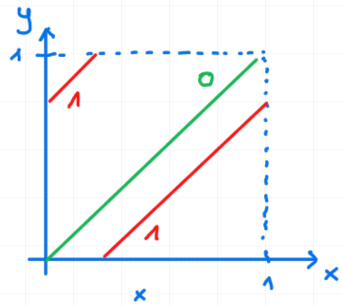
$\bar\gamma_n \in \Gamma(\mu,\nu)$, up to subsequences, $\bar\gamma_n \longrightarrow \bar\gamma$ narrowly.

So, $\forall n_0$

$$\int c_{n_0}\, d\bar\gamma = \lim_{n\to\infty} \int c_{n_0}\, d\bar\gamma_n \leq \underbrace{\liminf_{n\to\infty} \int c_n\, d\bar\gamma_n}_{\text{independent of } n_0}.$$

Take sup in n_0 and use monotone convergence theorem:

$$(KP) \leq \int c\, d\bar\gamma \leq \liminf_{n\to\infty} \int c_n\, d\bar\gamma_n \leq (DP).$$



<u>Example</u>: Let $\alpha \in [0,1]\setminus\mathbb{Q}$. $\underbrace{c(x,y)}_{\text{is l.s.c. but not continuous}} = \begin{cases} 0 & \text{if } x=y \\ 1 & \text{if } y = x-\alpha \\ +\infty & \text{otherwise} \end{cases}$

$\mu = \nu = \mathcal{L}^1_{[0,1]}$

Then: (A) $T_0(x) = x$ is optimal in (KP)?

(B) $T_1(x) = \begin{cases} x-\alpha & \text{if } 1 > x > \alpha \\ x+1-\alpha & \text{if } 0 < x < \alpha \end{cases}$ is optimal in (KP)?

(C) $\mathrm{supp}\,(Id, T_0)_\# \mathcal{L}_{[0,1]}$ is c-cycl. monotone?

(D) $\mathrm{supp}\,(Id, T_1)_\# \mathcal{L}_{[0,1]}$ is c-cycl. monotone?

Real life problems involving optimal transport (Santambrogio's Section 1.7.3) (Villani, Old & New, Section 5).

① (KP): X = skills of the employees

μ = (number of) distribution of employees with skill x

Y = tasks to distribute

ν = distribution of tasks

$p(x,y)$ productivity of employee with skill x when performing task y.

Goal of the company: $\max \left\{ \int p(x,y)\, d\gamma \;:\; \gamma \in \Gamma(\mu,\nu) \right\}$.

② Kantorovich duality: (X,μ) distribution of bakeries

(Y,ν) distribution of coffee shops

Bakery x sells bread to coffee shop y at cost $c(x,y)$.

A consortium wants to organize the market.

It buys for price $\underbrace{\psi_0(x)}_{\text{price per unit}}$ all bread of x, sells to y at price $\psi_1(y)$:

$$\text{Maximum profit: } \max_{-\psi_0(x)+\psi_1(y) \le c(x,y)} \left\{ -\int \psi_0(x)\, d\mu(x) + \int \psi_1(y)\, d\nu(y) \right\} =$$

$$\underset{\text{theorem}}{=} \min \left\{ \int c(x,y)\, d\gamma(x,y) : \gamma \in \Gamma(\mu,\nu) \right\}.$$

Remark: Another (formal) proof of (KP) = (DP):

Lagrange multiplier: it is 0 if $\pi_1 \# \gamma = \mu$, $+\infty$ otherwise.

$$\inf_{\gamma\in\Gamma(\mu,\nu)} \int_{X\times Y} c(x,y)\, d\gamma = \inf_{\gamma \ge 0} \left\{ \int c\, d\gamma + \sup_{\varphi} \left(\int_{X\times Y} \varphi(x)\, d\gamma(x,y) - \int_X \varphi(x)\, d\mu(x) \right) + \sup_{\psi} \left(\int_{X\times Y} \psi(y)\, d\gamma(x,y) - \int_Y \psi(y)\, d\nu(y) \right) \right\}$$

$$= \inf_{\gamma} \sup_{\varphi,\psi} \left\{ \int c\, d\gamma + \int \varphi(x)\, d\gamma - \int \varphi(x)\, d\mu + \int \psi(y)\, d\gamma - \int \psi(y)\, d\nu \right\} = (*)$$

we can exchange:

Lemma (Sion's minimax theorem): X compact, convex topological space, Y convex subset of linear space, $f: X\times Y \to \mathbb{R}$ continuous, convex in x, concave in y; then:

$$\min_{x\in X} \sup_{y\in Y} f(x,y) = \sup_{y\in Y} \min_{x\in X} f(x,y).$$

$$\text{take } \gamma = \underbrace{M}_{+\infty}\delta_{\bar{x},\bar{y}} \qquad \begin{cases} \equiv 0 & \text{if } c(x,y)+\varphi(x)+\psi(y)\geq 0 \ \forall (x,y)\in X\times Y \\ \equiv -\infty & \text{if } \exists \bar{x},\bar{y} : c(\bar{x},\bar{y})+\varphi(\bar{x})+\psi(\bar{y})<0. \end{cases}$$

$$(*) = \sup_{\varphi,\psi}\left\{-\int\varphi\,d\mu - \int\psi\,d\nu + \inf_{\gamma\geq 0}\int_{X\times Y}[c(x,y)+\varphi(x)+\psi(y)]\,d\gamma\right\}$$

$$= \sup_{\varphi(x)+\psi(y)+c(x,y)\geq 0} \int -\varphi\,d\mu + \int -\psi\,d\nu.$$

Exercise: Let $c(x,y) = \frac{1}{2}|x-y|^2$, φ convex, μ-a.e. differentiable.

Let $\mu \in \mathcal{P}(\mathbb{R}^d)$. Is $\nabla\varphi$ optimal between μ and $(\nabla\varphi)_{\#}\mu$?

Graph $\nabla\varphi$ is closed and $\supseteq \operatorname{supp}(Id,\nabla\varphi)_{\#}\mu \ \forall\mu$.

$\hookrightarrow$ c-cycl. monotone $\equiv \partial\varphi$.

By a previous Corollary, plan supported on a c-cycl. monotone set is optimal.

Brenier's theorem: Let $X = Y = \mathbb{R}^d$, $c(x,y) = \frac{1}{2}|x-y|^2$, let $\mu,\nu\in\mathcal{P}(\mathbb{R}^d)$,

with $\int|x|^2\,d\mu + \int|y|^2\,d\nu < +\infty$, $\mu \ll \mathcal{L}^d$.

Then, there exists a <u>unique</u> optimizer $\bar{\gamma}$ in (KP).

In addition, $\exists\, T = \nabla\varphi$, φ convex, s.t. $\bar{\gamma} = (Id, T)_{\#}\mu$,

$T = \nabla\varphi$ is the unique optimizer in (MP).

<u>Idea of the proof</u>:

∃ Let $\bar{\gamma}$ be an optimizer in (KP). By Corollary, supp $\bar{\gamma}$ is c-cycl. monotone,

$\operatorname{supp}\bar{\gamma} \subseteq \partial\varphi$, φ convex. If $\varphi\in C^1 \Longrightarrow \operatorname{supp}\bar{\gamma}\subseteq \operatorname{graph}\nabla\varphi$.

We claim $\bar{\gamma} = (Id,\nabla\varphi)_{\#}\mu$: $\forall F$ measurable ≥ 0,

$$\int F(x,y)\,d\bar{\gamma}(x,y) = \int F(x,\nabla\varphi(x))\,d\bar{\gamma}(x,y) = \int F(x,\nabla\varphi(x))\,d\mu(x) =$$

$$= \int F(x,y)\,d(Id,\nabla\varphi)_{\#}\mu.$$

$\hookrightarrow F(x,y) = F(x,\nabla\varphi(x))$ a.e. on $\operatorname{supp}\bar{\gamma}\subseteq\operatorname{graph}\nabla\varphi$

! Let $\gamma_1\neq\gamma_2$ optimal in (KP).

$\Longrightarrow \frac{\gamma_1+\gamma_2}{2}$ also optimal $\Rightarrow \gamma_1, \gamma_2, \frac{\gamma_1+\gamma_2}{2}$ are supported on the graph of a convex function.

This is a contradiction, because the union of two (different) graphs is not a graph.